2014

(6th Semester)

MATHEMATICS

Paper : Math–361

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—1

1. State the fundamental theorem on homomorphism of groups. Hence, prove that if H is a normal subgroup of a group G, and K is a normal subgroup of G containing H, then

$$\frac{G}{K} \cong \left(\frac{G}{H}\right) \Big/ \left(\frac{K}{H}\right)$$

2+8=10

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2\. (a) Show that $a \to a^{-1}$ is an automorphism of a group G if and only if G is Abelian. 5

(b) Show that the multiplicative group $G = \{1, -1, i, -i\}$ is isomorphic to the group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. 5

UNIT—2

3\. (a) Prove that every finite integral domain is a field. 7

(b) If S is an ideal of a ring R with unity 1 and $1 \in S$, then show that $S = R$. 3

4\. (a) Show that an ideal S of a commutative ring R is a prime ideal if and only if the residue class R/S is an integral domain. 6

(b) Show that a commutative ring with unity is a field if it has no proper ideal. 4

UNIT—3

5\. (a) Show that every Euclidean ring is a PID. 6

(b) If R is a commutative ring, then show that

(i) $a/b,\ b/c \Rightarrow a/c$

(ii) $a/b,\ a/c \Rightarrow a/(b+c)$ 4

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6. (a) Let R be a Euclidean ring and let a be a non-zero non-unit element in R. Suppose that $a = p_1 p_2 \cdots p_m = q_1 q_2 \cdots q_n$, where p_i, $i = 1, 2, \cdots, m$ and q_j, $j = 1, 2, \cdots, n$ are prime elements of R. Then show that $m = n$ and each p_i is an associate of some q_j and each q_j is an associate of some p_i. 6

(b) Let D be an integral domain with unity element 1. Show that two non-zero elements $a, b \in D$ are associated if and only if a / b and b / a. 4

UNIT—4

7. (a) Consider the set S of vectors $\alpha = (a_1, a_2, \cdots, a_n)$ in R^n, where all α are such that a_3 is an integer. Is S a subspace of R^n? Justify your answer. 1+2=3

(b) If U and V are two subspaces of a finite dimensional vector space V, then show that

$$\dim (U + V) = \dim U + \dim V - \dim (U \cap V)$$ 7

8. (a) Is the subset of a linearly independent set of vectors linearly independent? Justify your answer. 3

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(b) Define basis of a finite dimensional vector space. Show that every linearly independent subset of a finitely generated vector space V is either a basis of V or can be extended to form a basis of V. 2+5=7

UNIT—5

9. (a) Let V and W be vector spaces over the same field F and let T be a linear transformation from V into W. If V is finite dimensional, then show that

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$ 6

(b) Let $T: R^3 \to R^3$ be the linear transformation whose matrix representation with respect to the basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of R^3 is

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$$

Find the matrix B of T relative to the ordered basis $\{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ of R^3. 4

10. (a) Show that similar matrices have the same characteristic polynomial. 3

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(b) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$. Find the matrix P such that $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. 7

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2014

(6th Semester)

MATHEMATICS

Paper : Math–361

(Modern Algebra)

(PART : A—OBJECTIVE)

(*Marks* : 25)

The figures in the margin indicate full marks for the questions

Answer **all** questions

SECTION—A

(*Marks* : 10)

Put a Tick (✓) mark against the correct answer in the brackets provided for it : $1\times10=10$

1. Which of the following statements is false?

(a) A subgroup H of a group G is normal if and only if $x^{-1}Hx = H$ ( )

(b) If H is a normal subgroup of G and K is a normal subgroup of H, then K is a normal subgroup of G ( )

(c) Arbitrary intersection of two normal subgroups is a normal subgroup ( )

(d) The center Z of a group G is normal subgroup of G ( )

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2. The necessary and sufficient condition for a homomorphism f of a group G with identity e into a group G' with kernel K to be an isomorphism of G into G' is that

(a) $K = \phi$ ( )

(b) $K = \{e\}$ ( )

(c) $K = G$ ( )

(d) $K = G'$ ( )

3. The necessary and sufficient conditions for a non-empty subset S of a ring R to be a subring are

(a) $a \in S,\ b \in S \Rightarrow a + b \in S$ & $ab \in S$ ( )

(b) $a \in S,\ b \in S \Rightarrow a + b \in S$ & $\frac{a}{b} \in S$ ( )

(c) $a \in S,\ b \in S \Rightarrow a - b \in S$ & $ab \in S$ ( )

(d) $a \in S,\ b \in S \Rightarrow a - b \in S$ & $\frac{a}{b} \in S$ ( )

4. The set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in I$, the set of integers is

(a) a left ideal in the ring R of all 2×2 matrices with elements as integers ( )

(b) a right ideal in the ring R of all 2×2 matrices with elements as integers ( )

(c) an ideal in the ring R of all 2×2 matrices with elements as integers ( )

(d) a subring and not an ideal in the ring R of all 2×2 matrices with elements as integers ( )

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5. Let a be a non-zero element in the Euclidean ring R, then a is a unit if

(a) $d(a) \neq d(1)$ ( )

(b) $d(a) = d(1)$ ( )

(c) $d(a) < d(1)$ ( )

(d) $d(a) > d(1)$ ( )

6. The associates of a non-zero element $a + ib$ of the ring of Gaussian integers $D = \{a + ib, a, b \in I\}$ are

(a) $a + ib,\ a - ib,\ -a + ib,\ -a - ib$ ( )

(b) $a + ib,\ -a - ib,\ b + ia,\ b - ia$ ( )

(c) $a + ib,\ -a - ib,\ -b - ia,\ b - ia$ ( )

(d) $a + ib,\ -a - ib,\ -b + ia,\ b - ia$ ( )

7. Which of the following set of vectors is linearly independent in $V_3(R)$?

(a) $\{(1, 2, 1), (3, 1, 5), (3, -4, 7)\}$ ( )

(b) $\{(2, -3, 1), (3, -1, 5), (1, -4, 3)\}$ ( )

(c) $\{(2, 1, 2), (8, 4, 8)\}$ ( )

(d) $\{(-1, 2, 1), (3, 0, -1), (-5, 4, 3)\}$ ( )

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8. Which of the following functions T from R^2 into R^2 is a linear transformation?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$ ( )

(b) $T(x_1, x_2) = (x_1^2, x_2)$ ( )

(c) $T(x_1, x_2) = (x_1 - x_2, 0)$ ( )

(d) $T(x_1, x_2) = (\sin x_1, x_2)$ ( )

9. A necessary and sufficient condition for a square matrix A of order n over a field F to be diagonisable is that

(a) A has exactly n linearly independent eigenvectors ( )

(b) A has exactly n linearly dependent eigenvectors ( )

(c) A has exactly $(n+1)$ linearly independent eigenvectors ( )

(d) A has exactly $(n+1)$ linearly dependent eigenvectors ( )

10. The eigenvalues of a real symmetric matrix are

(a) purely imaginary ( )

(b) purely imaginary or zero ( )

(c) all zero ( )

(d) all real ( )

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SECTION—B

(*Marks* : 15)

Answer the following questions : 3×5=15

1. If G is a group and H is a subgroup of index 2, then show that H is a normal subgroup of G.

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2. Show that a field has no proper ideals

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3. If f is a homomorphism of a ring R into a ring R' with kernel K, then show that K is an ideal of R.

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4. Show that if two vectors are linearly dependent, then one of them is a scalar multiple of the other.

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8. Show that two eigenvectors of a square matrix A over a field F corresponding to two distinct eigenvalues are linearly independent.

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