2016

(6th Semester)

MATHEMATICS

Paper : Math–361

(Modern Algebra)

Full Marks : 75

Time : 3 hours

(PART : B—DESCRIPTIVE)

(*Marks* : 50)

The figures in the margin indicate full marks for the questions

Answer **one** question from each Unit

UNIT—I

1. *(a)* Prove that the centre Z of a group is a normal subgroup of G. 5

(b) Let G be a group. Prove that the mapping $f : G \quad G$ given by $f(a) \quad a^{1} \quad a \quad G$ is an isomorphism if and only if G is Abelian. 5

2. The set $I(G)$ of all inner automorphisms of a group G is a normal subgroup of $A(G)$, the group of all automorphisms of G. Prove that

$$I(G) \quad G / Z(G)$$

$Z(G)$ is the centre of G. 10

UNIT—II

3. *(a)* Prove that every field is an integral domain. 5

(b) Show that the ring Z_p of integer modulo p is a field if and only if p is a prime. 5

4. *(a)* Prove that the characteristic of an integral domain is either 0 or a prime number. 5

(b) Let R be a commutative ring. Prove that an ideal P of R is a prime ideal if and only if R / P is an integral domain. 5

UNIT—III

5. Prove that every homomorphic image of a ring R is isomorphic to some quotient ring of R. 10

6. *(a)* Prove that in a UFD, every prime element is irreducible. 5

(b) Prove that every Euclidean domain has a unit element. 5

UNIT—IV

7. *(a)* If U and V are two subspaces of a finite dimensional vector space V, then show that

$$\dim(U \quad V) \quad \dim U \quad \dim V \quad \dim(U \quad V)$$ 7

(b) Prove that the set of vectors

$$\{(1,\ 2,\ 3),\ (2,\ 1,\ 2),\ (2,\ 2,\ 1)\}$$

is linearly independent in R^3. 3

8. *(a)* If W be a subspace of a finite dimensional vector space $V(F)$, then prove that

$$\dim \frac{V}{W} \quad \dim V \quad \dim W$$ 6

(b) Let $T : R^3 \quad R^3$ be the linear transformation defined by

$$T(x,\ y,\ z) \quad (3x,\ x \quad y,\ 2x \quad y \quad z)$$

Show that T is a homomorphism. 4

UNIT—V

9. *(a)* Let V and W be vector spaces over the same field F and also let T be a linear transformation from V into W. If V is finite dimensional, then show that

$$\text{rank } (T) + \text{nullity } (T) = \dim V$$ 6

(b) Let $T : R^3 \quad R^3$ be a linear transformation defined by

$$T(x,\ y,\ z) \quad (3x \quad z,\quad 2x \quad y,\quad x \quad 2y \quad 4z)$$

Compute the matrix A of T with respect to the standard basis of R^3. 4

10. Let $T : R^3 \quad R^3$ be a linear transformation whose matrix with respect to the standard basis of R^3 is

$$A \quad \begin{array}{ccc} \cos & \sin & 0 \\ \sin & \cos & 0 \\ 0 & 0 & 1 \end{array}$$

Find the matrix B of T with respect to the standard basis $\{(1,\ 0,\ 0),\ (1,\ 1,\ 0),\ (1,\ 1,\ 1)\}$ of R^3. Also find a non-singular matrix P such that $PAP^{\ 1} \quad B$. 10

★★★

Subject Code : MATH/VI/09

Booklet No. A

Date Stamp ....

....

To be filled in by the Candidate

DEGREE 6th Semester
(Arts / Science / Commerce / ....) Exam., **2016**

Subject ....
Paper ....

To be filled in by the Candidate

DEGREE 6th Semester
(Arts / Science / Commerce / ....) Exam., **2016**

Roll No. ....
Regn. No. ....

Subject ....
Paper ....

Descriptive Type
Booklet No. B ....

INSTRUCTIONS TO CANDIDATES

1. **The Booklet No. of this script should be quoted in the answer script meant for descriptive type questions and vice versa.**
2. **This paper should be ANSWERED FIRST and submitted within 1 (one) Hour of the commencement of the Examination.**
3. **While answering the questions of this booklet, any cutting, erasing, over-writing or furnishing more than one answer is prohibited. Any rough work, if required, should be done only on the main Answer Book. Instructions given in each question should be followed for answering that question only.**

Signature of Scrutiniser(s)

Signature of Examiner(s)

Signature of Invigilator(s)

/340

2016

(6th Semester)

MATHEMATICS

Paper : Math–361

(Modern Algebra)

(PART : A—OBJECTIVE)

(*Marks* : 25)

Answer **all** questions

SECTION—A

(*Marks* : 10)

Each question carries 1 mark

Put a Tick (✓) mark against the correct answer in the brackets provided :

1. Which of the following statements is false?

(a) A subgroup H of a group G is normal if and only if each left coset of H in G is a right coset of H in G ( )

(b) Every subgroup of an Abelian group is normal ( )

(c) If H is a normal subgroup of G and K is a subgroup of G, then $H \quad K$ is a normal subgroup of K ( )

(d) A subgroup H of G is normal if and only if $ab \quad H$ implies that $ba \quad H$, a, $b \quad G$ ( )

2. If f is a homomorphism of G into G, then K is the kernel of f if

(a) $k \quad \{x \quad G : f(x) \quad e\}$ ( )

(b) $k \quad \{x \quad G : f(x) \quad e\}$ ( )

(c) $k \quad \{x \quad G : f(x) \quad 0\}$ ( )

(d) $k \quad \{x \quad G : f(x) \quad x\}$ ( )

3. The necessary and sufficient conditions for a non-empty subset S of a ring R to be a subring are

(a) $a \quad b \quad S$ and $\frac{a}{b} \quad S$ for all $a, b \quad S$ ( )

(b) $a \quad b \quad S$ and $ab \quad S$ for all $a, b \quad S$ ( )

(c) $a \quad b \quad S$ and $\frac{a}{b} \quad S$ for all $a, b \quad S$ ( )

(d) $a \quad b \quad S$ and $ab \quad S$ for all $a, b \quad S$ ( )

4. The proper ideals of Z_{12} are 2 , 3 , 4 and 6 , then the maximal ideals are

(a) 2 and 4 ( )

(b) 2 and 6 ( )

(c) 2 and 3 ( )

(d) 4 and 6 ( )

5. The units in Z_8 $\{0, 1, 2, 3, 4, 5, 6, 7\}$ modulo 8 are

(a) 0, 2, 4, 6 ( )

(b) 1, 3, 5, 6 ( )

(c) 1, 3, 5, 7 ( )

(d) 4, 5, 6, 7 ( )

6. The associates of a non-zero element $a \quad ib$ of the ring of Gaussian integers $D \quad \{a \quad ib, a, b \quad I\}$ are

(a) $a \quad ib, \quad a \quad ib, \quad b \quad ia, b \quad ia$ ( )

(b) $a \quad ib, \quad a \quad ib, \quad b \quad ia, b \quad ia$ ( )

(c) $a \quad ib, \quad a \quad ib, b \quad ia, b \quad ia$ ( )

(d) $a \quad ib, a \quad ib, \quad a \quad ib, \quad a \quad ib$ ( )

7. Which of the following statements is false?

If A and B are subspaces of V, then

(a) $A \quad B$ is a subspace of V ( )

(b) A is a subspace of $A \quad B$ ( )

(c) B is a subspace of $A \quad B$ ( )

(d) every element of $A \quad B$ can be uniquely written in the form $a \quad b$, where $a \quad A$, $b \quad B$ and $A \quad B \quad \{0\}$ ( )

8. For the vector space $V_3(F)$, which set is not a basis?

(a) (1, 0, 0), (1, 1, 0), (1, 1, 1) ( )

(b) (1, 0, 0), (0, 1, 0), (0, 0, 1) ( )

(c) (1, 0, 0), (0, 1, 0), (1, 1, 1) ( )

(d) (1, 0, 1), (1, 0, 0), (0, 0, 1) ( )

9. If $A \quad \begin{matrix} 0 & 1 \\ 4 & 4 \end{matrix}$, then the eigenvalues of A are

(a) 0, 4 ( )

(b) 2, 2 ( )

(c) 1, 4 ( )

(d) 1, 4 ( )

10. The eigenvalues of a real skew-symmetric matrix are

(a) purely imaginary ( )

(b) all zero ( )

(c) purely imaginary or zero ( )

(d) all real ( )

SECTION—B

(*Marks* : 15)

Each question carries 3 marks

1. If H is a normal subgroup of G and K is a subgroup of G, then prove that H K is a normal subgroup of K.

2. Prove that the centre of a ring R is a subring of R.

3. Show that $1 \quad i$ is an irreducible element in $Z[i]$.

4. Prove that the intersection of any two subspaces W_1 and W_2 of a vector space is again a subspace of $V(F)$.

5. Prove that similar matrices have the same eigenvalues.

★★★