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(NEP-2020)

(3rd Semester)

MATHEMATICS (MAJOR)

(Modern Algebra—I)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(Marks : 10)

Tick the correct answer in the boxes provided :

1×10=10

1. The inverse of the element a in the group of all positive rational numbers under the composition $a \ b \ \frac{ab}{2}$ is

(a) 2

(b) $a / 4$

(c) $4 / a$

(d) $4a$

2. The number of binary compositions on a finite set A having n elements is

(a) 2^{n^2}

(b) n^{n^2}

(c) n^n

(d) n^2

3. In $(Z_6, +_6)$, where $Z_6 = \{0, 1, 2, 3, 4, 5\}$ and $+_6$ is the addition modulo 6, the order of 4 is

(a) 2

(b) 3

(c) 4

(d) 5

4. If a and b are any two elements of a group G and H is any subgroup of G , then $Ha = Hb$ if and only if

(a) $ab \in H$

(b) $a^{-1}b \in H$

(c) $ab^{-1} \in H$

(d) $(ab)^{-1} \in H$

5. Let $T = \{1, i, j, k\}$ be a quaternion group. Then

- (a) T is non-Abelian with order 4
- (b) T is non-Abelian with order 8
- (c) T is Abelian with order 4
- (d) T is Abelian with order 8

6. Which of the following is **incorrect**?

- (a) A group of order 1 is Abelian.
- (b) Every commutative group is Abelian.
- (c) In a group (G, \cdot) , where \cdot denotes ordinary multiplication, a^{-1} and b are commutative if a and b are commutative $a, b \in G$.
- (d) If every element of a group G is its own inverse, then G is non-Abelian.

7. The order of a finite Abelian group $(G, +_4)$, where $G = \{0, 1, 2, 3\}$ and $+_4$ as addition modulo 4, is

- (a) 6
- (b) 5
- (c) 4
- (d) 3

8. Let a be any element of an Abelian group G such that $ax = xa = x \in G$. Then

(a) $N(a) = G$

(b) $N(a) = \{a\}$

(c) $N(a) = \{e\}$

(d) $N(a) = \{a, a^{-1}\}$

where $N(a)$ is normalizer of an element of G .

9. The generator of the cyclic group $(\mathbb{Z}/12\mathbb{Z}, +)$ is

(a) 1 and 5

(b) 1 only

(c) 5 only

(d) $1, 5$ and 7

10. The inverse of the product of $(1\ 2)$ and $(2\ 3)$ of the symmetric group S_3 is

(a) $(1\ 2\ 3)$

(b) (1)

(c) $(1\ 3)$

(d) $(1\ 3\ 2)$

(SECTION : B—SHORT ANSWERS)

(Marks : 15)

Answer *five* questions, taking at least *one* from each Unit :

3×5=15

UNIT—I

1. In the set of real numbers, show that the composition $x \circ y = x^y$ for all $x, y \in \mathbb{R}$ is not associative.
2. Prove that the inverse of the product of two elements of a group G is the product of the inverse taken in the reverse order.

UNIT—II

3. Prove that the intersection of two subgroups of a group G is a subgroup of G .
4. Prove that any two right cosets of a subgroup are either disjoint or identical.

UNIT—III

5. Prove that a group G is an Abelian if $a^2 = e \quad \forall a \in G$, where e is the identity element of G .
6. Show that the symmetric group S_n is Abelian for $n = 2$ and non-Abelian other than $n = 2$.

UNIT—IV

7. Find the remainder when 45^{31} is divided by 33.
8. If a finite group of order n contains an element of order n , then prove that the group must be cyclic.

(SECTION : C—DESCRIPTIVE)

(Marks : 50)

Answer *five* questions, taking at least *one* from each Unit :

10×5=50

UNIT—I

1. (a) Let A be the set of fourth root of unity, i.e., $A = \{1, -1, i, -i\}$. Then show that the algebraic structure (A, \cdot) is a group, where \cdot is an ordinary multiplication. 5
- (b) Prove that the set $G = \{1, 2, 3, 4, 5, 6\}$ is a group with respect to multiplication modulo 7, using group table. 5
2. (a) Prove that the identity element in a group is unique. Also prove that the inverse of each element of a group is unique. 5
- (b) If a and b are any two elements of a group G , then show that the equations $ax = b$ and $ya = b$ have unique solutions in G . 5

UNIT—II

3. (a) Prove that the necessary and sufficient condition for a non-empty subset K of a group G to be a subgroup is that $r \in K, s \in K \implies rs^{-1} \in K$; where s^{-1} is the inverse of s in G . 5
- (b) If H and K are two subgroups of a group G , then show that HK is a subgroup of G if and only if $HK = KH$. 5
4. (a) Prove that the order of every element of a finite group is finite and is less than or equal to the order of the group. 5
- (b) State and prove Lagrange's theorem on the order of a group. 5

UNIT—III

5. Prove that—

(a) the set of all n th roots of unity forms a finite Abelian group of order n with respect to multiplication;

(b) G is Abelian if it is a group with $(ab)^m = a^m b^m$ for three consecutive integers $m, a, b \in G$. 7+3=10

6. (a) Show that if G is an Abelian group, then for any integer n such that

$$(ab)^n = a^n b^n \quad a, b \in G \quad 5$$

(b) Prove that a group G is Abelian if and only if

$$(pq)^{-1} = p^{-1} q^{-1} \quad p, q \in G \quad 5$$

UNIT—IV

7. (a) Show that every finite group is isomorphic to a permutation group. 7

(b) Find the order of permutation of $(23)(134)(13)$ in the symmetric group S_4 . 3

8. Prove that—

(a) for any set S , $A(S)$ is a group with respect to composition mapping where $A(S)$ denotes the set of all one to one mapping of S onto itself;

(b) the normalizer of a group G is a subgroup of G . 5+5=10
