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(NEP—2020)

(3rd Semester)

MATHEMATICS (MAJOR)

(Modern Algebra—I)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Put a Tick (✓) mark against the correct answer in the brackets provided : 1×10=10

1. The inverse of the element a of the group of all integers under the operation defined by $a \quad b \quad a \quad b \quad 1$ is

(a) $2 \quad a$ ( )

(b) $2 \quad a$ ( )

(c) $2 \quad a$ ( )

(d) $a \quad 2$ ( )

2. The number of commutative binary operations on a finite set A having n elements is

(a) n^2 ( )

(b) $\frac{n^2}{2}$ ( )

(c) $\frac{n^2 \quad 1}{2}$ ( )

(d) n^{n^2} ( )

3. In multiplicative group of rational numbers, the order of 7 is

(a) 0 ( )

(b) 1 ( )

(c) 2 ( )

(d) infinite ( )

4. Let G be the additive group of integers and $H \quad \{3x : x \quad G\}$. Then the number of distinct right cosets is

(a) 2 ( )

(b) 3 ( )

(c) 1 ( )

(d) infinite ( )

5. The set of all permutations on the set $\{1, 2, 3\}$ is the group of order

(a) 3 ( )

(b) 1 ( )

(c) 6 ( )

(d) 9 ( )

6. Let $T \quad \{ 1, \quad i, \quad j, \quad k\}$ be a quaternion group. Then

(a) T is non-Abelian group of order 4 ( )

(b) T is non-Abelian group of order 8 ( )

(c) T is Abelian group of order 4 ( )

(d) T is Abelian group of order 8 ( )

7. The number of generators of the cyclic group of order 8 is

(a) 4 ( )

(b) 3 ( )

(c) 8 ( )

(d) 7 ( )

8. The special linear group $SL(n, R)$ is of all $n \quad n$ matrices with determinant equal to

(a) 0 ( )

(b) 1 ( )

(c) –1 ( )

(d) None of the above ( )

9. The generator(s) of the group $G \quad (\{0, 1, 2, 3, 4, 5\} \quad _6)$ are/is

(a) 0 and 1 ( )

(b) 1 and 5 ( )

(c) 0 and 5 ( )

(d) only 5 ( )

10. In S_3, $\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}$ is a cycle of length

(a) 3 ( )

(b) 2 ( )

(c) 1 ( )

(d) None of the above ( )

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—1

1. Show that the matrix multiplication is associative but not commutative binary operation on $M_2(R)$, the set of all 2   2 matrices over R.

2. Prove that the inverse of each element of a group is unique.

UNIT—2

3. Show that the union of two subgroups is not necessarily a subgroup.

4. If the elements of a group G is of order n, then show that a^m e if and only if n is a divisor of m.

UNIT—3

5. If a and b are any two elements of a group G, then show that $(ab)^2$ a^2b^2 if and only if G is abelian.

6. In a dihedral group of D_3, calculate the total number of subgroups and normal subgroups of D_3.

UNIT—4

7. If a is a generator of a cyclic group G, then show that $a^{\ \ 1}$ is also a generator of G.

8. Find the remainder when 99^{20} is divided by 25.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, selecting at least *one* from each Unit : 10×5=50

UNIT—1

1. *(a)* Prove that the set $G\ \{0, 1, 2, 3, 4, 5\}$ is a finite Abelian group with respect to addition modulo 6. 5

(b) Show that the set $G\ \{1,\ ,\ ^2\}$, where is an imaginary cube root of unity, is a group with respect to multiplication. 5

2. *(a)* If a and b are any two elements of a group G, then show that the equations $ax\ \ b$ and $ya\ \ b$ have unique solution in G. 5

(b) Prove that the inverse of the product of two elements of a group G is the product of inverse taken in reverse order. 5

UNIT—2

3. *(a)* Let G be the additive group of integers. Then prove that the set of all multiples of integers by a fixed integer m is a subgroup of G. 5

(b) Show that the union of two subgroups is a subgroup if and only if one is contained in the other. 5

4. *(a)* Prove that the order of an element of a group is the same as that of its inverse. 4

(b) Prove that the order of each subgroup of a finite group is a divisor of the group. 6

UNIT—3

5. *(a)* If a group G has four elements, then show that it must be abelian. 5

(b) Show that if every element of a group G is its own inverse, then G is abelian. 5

6. *(a)* Define special linear group $SL(n, R)$. Show that $SL(n, R)$ is a non-Abelian group under matrix multiplication for n 2. 2+3=5

(b) Prove that a group G is abelian if every element of G except identity element is of order 2. 5

UNIT—4

7. *(a)* Define cyclic group. Prove that every infinite cyclic group has two and only two generators. 1+4=5

(b) Prove that the normalizer $N(a)$ of a G is a subgroup of G. 5

8. *(a)* State and prove Cayley's theorem. 6

(b) Write down all the permutations on four symbols 1, 2, 3, 4. Which of these permutations are even? 4

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