MATH300 **Student's Copy**

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(NEP—2020)

(5th Semester)

MATHEMATICS (MAJOR1)

(Advanced Calculus)

Full Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

(SECTION : A—OBJECTIVE)

(*Marks* : 10)

Tick ☑ the correct answer in the boxes provided : 1×10=10

1. Let P be a refinement of a partition P, then for a bounded function f

(a) $L(P\ ,\ f)\quad L(P,\ f)$ ☐

(b) $U(P\ ,\ f)\quad U(P,\ f)$ ☐

(c) $U(P\ ,\ f)\quad L(P,\ f)$ ☐

(d) $L(P\ ,\ f)\quad L(P,\ f)$ ☐

2. If $f(x) \quad x$ over [0, 1] and

$$P \quad 0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n \quad 1}{n}, \frac{n}{n} \quad 1$$

be the partition, then the value of the lower Riemann integral ${}^{1}_{\underline{0}} f\,dx$ is

(a) $\frac{1}{2}$ ☐

(b) $\frac{1}{3}$ ☐

(c) $\frac{1}{2}$ ☐

(d) None of the above ☐

3. If f and g be two positive functions such that $f(x) \quad g(x) \quad x \quad [a, b]$, then

(a) ${}^{b}_{a} g\,dx$ converges if ${}^{b}_{a} f\,dx$ converges ☐

(b) ${}^{b}_{a} f\,dx$ diverges if ${}^{b}_{a} g\,dx$ diverges ☐

(c) ${}^{b}_{a} f\,dx$ converges if ${}^{b}_{a} g\,dx$ converges ☐

(d) None of the above ☐

4. The improper integral ${}_{a} \frac{1}{x^n}\,dx$ converges if and only if

(a) $n \quad 1$ ☐

(b) $n \quad 1$ ☐

(c) $n \quad 1$ ☐

(d) $n \quad 1$ ☐

5. Uniformly convergent improper integral of a continuous function is

(a) not continuous ☐

(b) may be continuous ☐

(c) itself continuous ☐

(d) differentiable ☐

6. The value of the improper integral $_0 \dfrac{dx}{a \quad b\cos x}$ if a is positive and $|b| \quad a$ is

(a) $\dfrac{2}{(a^2 \quad b^2)^{1/2}}$ ☐

(b) $\dfrac{2}{(a^2 \quad b^2)^{3/2}}$ ☐

(c) $\dfrac{}{(a^2 \quad b^2)^{3/2}}$ ☐

(d) $\dfrac{}{(a^2 \quad b^2)^{1/2}}$ ☐

7. The value of the integral $_C \dfrac{dx}{x \quad y}$, where C is the curve $x \quad at^2$, $y \quad 2at$, $0 \quad t \quad 2$ is

(a) $\dfrac{1}{6}$ ☐

(b) $\log 4$ ☐

(c) $\dfrac{1}{6}$ ☐

(d) $\log 2$ ☐

8. The value of $_C 4x^3\, ds$ where C is the line segment from $(\quad 2, \quad 1)$ to $(1, \; 2)$ is

(a) $15\sqrt{2}$ ☐

(b) $15\sqrt{2}$ ☐

(c) $15\sqrt{3}$ ☐

(d) $25\sqrt{2}$ ☐

9. The sequence $\{f_n\}$ of continuous function is uniformly convergent to a function f on $[a, b]$. Then f is also

(a) uniformly convergent ☐

(b) continuous ☐

(c) integrable ☐

(d) differentiable ☐

10. By M_n-test, the sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if

(a) M_n 0 as n 0 ☐

(b) M_n as n 0 ☐

(c) M_n 0 as n ☐

(d) None of the above ☐

(SECTION : B—SHORT ANSWERS)

(*Marks* : 15)

Answer *five* questions, taking at least *one* from each Unit : 3×5=15

UNIT—I

1. Show that $f(x)$ x^2 is R-integrable on any interval $[0, k]$.

2. If a function f is monotonic on $[a, b]$, then prove that it is R-integrable over $[a, b]$.

UNIT—II

3. Examine the convergence of the function

$$_0 \; \frac{x \tan^{\;1} x}{(1 \quad x^4)^{1/3}} dx$$

4. Establish the uniform convergence of the improper integral

$$_0 \; \frac{y}{x^2 \quad y^2} dx$$

with $(0 \quad c \quad y \quad d)$.

UNIT—III

5. Show that

$$_C[(x \quad y)^3 dx \quad (x \quad y)^3 dy] \quad 3 \quad a^4$$

taken along the circle $x^2 \quad y^2 \quad a^2$ in the counterclockwise sense.

6. Evaluate the double integral

$$_{/2} \; _0^{x^2} \frac{1}{x} \cos \frac{y}{x} \, dy \, dx$$

UNIT—IV

7. Prove that the sequence $f_n(x) \quad nxe^{\quad nx^2}$ is pointwise convergent but not uniformly on $[0, \quad [$ by evaluating the pointwise limit.

8. Show by M_n-test, the sequence of function

$$f_n(x) \quad \frac{x}{1 \quad nx^2} \quad x \quad \mathbb{R}$$

converges uniformly on any closed interval.

(SECTION : C—DESCRIPTIVE)

(*Marks* : 50)

Answer *five* questions, taking at least *one* from each Unit : 10×5=50

UNIT—I

1. *(a)* Prove that a necessary and sufficient condition for the integrability of bounded function f is that to every $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partition P of $[a, b]$ with norm $\mu(P) < \delta$. 5

(b) Show that $f(x) = 3x + 1$ is R-integrable on [1, 2] and prove that

$$\int_1^2 f(x)\,dx = \frac{11}{2}$$

5

2. *(a)* State Darboux's theorem and apply it to show that if f is bounded and integrable on $[a, b]$, then to every $\epsilon > 0$, $\exists\ \delta > 0$ such that for every partition $P = \{a = x_0, x_1, x_2, \cdots\cdots, x_n = b\}$ with norm $\mu(P) < \delta$ and for every choice of $t_r \in [x_{r-1}, x_r]$

$$\left| \sum_{r=1}^{n} f(t_r)(x_r - x_{r-1}) - \int_a^b f(x)\,dx \right| < \epsilon$$

1+4=5

(b) Compute the value of $\int_{-1}^{1} f\,dx$, where $f(x) = |x|$ by dividing the interval $[-1, 1]$ into $2n$-equal sub-intervals. 5

UNIT—II

3. *(a)* By Cauchy's test, show that $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent. 5

(b) Prove that the integral $\int_0^\infty x^{n-1} e^{-x}\,dx$ is convergent if and only if $n > 0$. 5

4. *(a)* Let $(y) \quad {}_a^b f(x, y)\,dx,$ is continuous and f_y also exists and continuous in $[a, b; c, d]$, then prove that is derivable and

$$(y) \quad {}_a^b f_y(x, y)\,dx \quad y \quad [c, d]$$ 5

(b) Test the uniform convergence of the convergent improper integral

$$_0 \, e \quad {}^{x^2} \cos yx\,dx$$

in (,). 5

UNIT—III

5. *(a)* Evaluate the integral

$$\begin{matrix} 3 & 9 \\ 0 & x^2 \end{matrix} \; x^3 e^{y^3}\,dy\,dx$$

by reversing the order of integration with a rough figure. 5

(b) Show that

$$\begin{matrix} 1 \\ 0 \end{matrix} \; \begin{matrix} 1 \\ 0 \end{matrix} \frac{x^2 \quad y^2}{x^2 \quad y^2}\,dy \quad dx \qquad \begin{matrix} 1 \\ 0 \end{matrix} \; \begin{matrix} 1 \\ 0 \end{matrix} \frac{x^2 \quad y^2}{x^2 \quad y^2}\,dx \quad dy$$

i.e., change in order of integration is permissible. 5

6. *(a)* Change the order of integration in the integral

$$\begin{matrix} 1 \\ 0 \end{matrix} dx \quad \begin{matrix} \sqrt{1 \quad x^2} \\ 0 \end{matrix} \; \frac{e^y dy}{(1 \quad e^y)\sqrt{1 \quad x^2 \quad y^2}}$$

and hence evaluate it with a rough figure. 5

(b) Prove that

$$\begin{matrix} 1 \\ 0 \end{matrix} \; \begin{matrix} 1 \\ 0 \end{matrix} \frac{(x \quad y)}{(x \quad y)^3}\,dy \quad dx \qquad \begin{matrix} 1 \\ 0 \end{matrix} \; \begin{matrix} 1 \\ 0 \end{matrix} \frac{(x \quad y)}{(x \quad y)^3}\,dx \quad dy$$

i.e., change in order of integration is not permissible. 5

UNIT—IV

7. *(a)* Show that $f_n(x) \quad e^{\quad nx}$ is pointwise but not uniform convergent in $[0. \quad [$. Also show that the convergence is uniform in $[k, \quad [$, where k is any positive number. 5

(b) Show that a sequence of function $\{f_n(x)\}$ defined on $[a, b]$ is said to converge uniformly to f on $[a, b]$ if and only if to each $\quad 0$ and $\quad x \quad [a, b]$, there exists an integer m such that

$$|f_{n \quad p}(x) \quad f_n(x)| \qquad n \quad m, \ p \quad 1$$ 5

8. *(a)* Show that the series

$$\quad_{n \quad 1} \frac{x}{n^3 \quad n^4 x^2}$$

is uniformly convergent for all values of x and that the derivative of the sum with respect to x is given by term-by-term differentiation. 5

(b) If a sequence $\{f_n\}$ converges uniformly to f on $x \quad [a, b]$ and let f_n be integrable $\quad n$, then prove that f is integrable and

$$\quad_a^b f(x)dx \quad \lim_{n} \quad_a^b f_n(x)\,dx$$ 5

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